The theory of functional equations and inequalities and its applications

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February 10 (Thursday)
Functional equations in several variables

Roman Ger

Ideal stability method of solving functional equations

Péter Erdei

Conditional equations for additive functions
(joint work with Zoltán Boros)

Let $S = \{ (x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1 \}$. The following problems were formulated by Walter Benz in 1988 and 1989 (during the 26^{th} and 27^{th} International Symposia on Functional Equations).

Problem 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive function satisfying

$$xf(x) + yf(y) = 0$$

for every $(x, y) \in S$. Does it imply that f is a derivation?

Problem 2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive function satisfying

$$yf(x) = xf(y)$$

for every $(x, y) \in S$. Does it imply that f is linear?

We establish affirmative answers to both questions of Benz.

Tomasz Szostok

Orlicz functions and functional equations

A function $\varphi : \mathbb{R} \to \mathbb{R}$ is called an Orlicz function if and only if it is convex, even, vanishes at zero and is different from 0. Such functions are used to construct certain Banach spaces which are called Orlicz spaces. In order to obtain some geometrical properties of the Orlicz space generated by φ one has to assume some properties of φ. One of such properties is the following

$$\bigwedge_{a\in[0,\infty)} \bigvee_{\gamma(a)\in\left(0,\frac{1}{2}\right)} \bigwedge_{x,y\in[0,\infty),x\le ay} \varphi\left(\frac{x+y}{2}\right) \le \gamma(a)[\varphi(x)+\varphi(y)].$$

We study properties of this kind and also some related functional equations and inequalities.

Tomasz Kochanek

Solutions of the Gołąb–Schinzel equation which are continuous at zero

There are the known results of P. Javor, N. Brillouët–J. Dhombres and K. Baron which establish forms of continuous solutions of the Gołąb–Schinzel equation

$$f(x+f(x)y) = f(x)f(y)$$

which are real or complex-valued and defined on a linear topological Hausdorff space X. It occurs that it is enough to assume continuity at zero (or at any point that a given function does not vanish in) of a given function to obtain the same assertions as in the above mentioned theorems. This fact can be easily deduced from a stronger theorem of J. Brzdęk [*Bounded solutions of the Gołąb–Schinzel equation*, Aequationes Math. 59 (2000) 248-254] in the case when X is a Fréchet space. However, we present an independent and direct proof of that fact for an arbitrary X. We also present a counterexample showing that the assumption about a given point of continuity is essential.

Fruzsina Mészáros

A functional equation satisfied almost everywhere relating to characterization problems

(joint work with Károly Lajkó)

Functional equations are used in the characterization of joint distributions by means of conditional distributions. I look for the joint density function in special cases of the density functions for what I investigate the general measurable solution of functional equations satisfied almost everywhere.

Problem 1. What is the general measurable solution of the functional equation

$$g_1\left(\frac{x-m_1y-c_1}{\lambda_1\left(y+a_1\right)}\right) f_Y\left(y\right) = g_2\left(\frac{y-m_2x-c_2}{\lambda_2\left(x+a_2\right)}\right) f_X\left(x\right),$$

satisfied almost all $(x,y)\in D\subset\mathbb{R}^2_+$, where

$$D=\left\{(x,y)\in\mathbb{R}^2 \mid x+a_2>0, y+a_1>0\right\},$$

and $\lambda_1,\lambda_2>0$?

Problem 2. Let D_1, D_2 be the following sets:

$$D_1=\mathbb{R}\setminus\left\{-b_1,-b_2\right\}, D_2=\mathbb{R}\setminus\left\{-a_1,-a_2\right\}.$$

What is the general measurable solution of the functional equation

$$g_1\left[\left(y+a_1\right)x+b_1y+c_1\right]f_Y\left(y\right)=g_2\left[\left(x+b_2\right)y+a_2x+c_2\right]f_X\left(x\right),$$

satisfied almost all $(x,y)\in D=D_1\times D_2$?

I shall present the general measurable solutions of these functional equations.

Włodzimierz Fechner

On functional equations and inequalities connected with additive and quadratic functionals

Let $(G,+)$ be a group and $(X,\|\cdot\|)$ a normed linear space. We consider two functional inequalities

$$\|f(x)+f(y)\| \leq \|f(x+y)\|, \tag{1}$$
$$\|2g(x)+2g(y)-g(x-y)\| \leq \|g(x+y)\|, \tag{2}$$

where $f\colon G\to X$ and $g\colon G\to X$ are unknown mappings. We will provide conditions under which f and g have to be: additive and quadratic, respectively. Moreover, we will discuss the Hyers-Ulam stability of (1) and (2).

February 11 (Friday)
Iterative functional equations

Karol Baron

Methods of iterative functional equations theory

Agnieszka Klama

Random-valued functions and iterative functional equations
(work of Karol Baron and Witold Jarczyk)

Given a random-valued function $f:[0,1]\times\Omega\to[0,1]$ on a probability space $(\Omega,\mathcal{A},P)$, we consider bounded solutions $\psi:[0,1]\to\mathbb{R}$ of the inequality

$$\psi(x)\leq\int_\Omega\psi(f(x,\omega))\,dP(\omega)$$

and a uniqueness-type problem for bounded solutions ϕ of equations of the type

$$\phi(x)=h(x,\phi\circ f(x,\cdot)).$$

Analogues for $f:\mathbb{R}\times\Omega\to\mathbb{R}$ of the form $f(x,\omega)=x+\xi(\omega)$ are proved. Some particular cases are studied in more details, especially those where the probability space under considerations is simply the set of positive integers.

Pál Burai

Functional equation of Domsta and Matkowski

We are going to prove an extension theorem for a functional equation studied by Domsta and Matkowski.

Pola Siwek

Solutions with big graph
of homogeneous functional equations in a single variable
(work of Lech Bartłomiejczyk)

We obtain a general result on the existence of solutions with big graph of functional equations of the form

$$g(x, \phi(x), \phi(f_1(x)), \phi(f_2(x)), \ldots) = 0\,,$$

assuming a homogeneity condition on the given function g, and we apply it to some particular equations, both classical and recently considered in the theory of functional equations.

Dariusz Sokołowski

On connections between linear functional equations
and real roots of their characteristic equations

Inspired by R.O. Davies and A.J. Ostaszewski [J. Math. Anal. Appl. 247 (2000), 608-626] we investigate how the existence of solutions φ having a constant sign in a vicinity of infinity of the equation

$$\varphi(x) = \int_S \varphi(x + M(s))\sigma(ds) \tag{1}$$

depends on real roots λ of its characteristic equation

$$\int_S e^{\lambda M(s)}\sigma(ds) = 1. \tag{2}$$

Here (S, Σ, σ) is a measure space with a finite measure σ and $M\colon\ S \to \mathbb{R}$ is a Σ -measurable bounded function. By a *solution* of (1) we mean a Borel measurable real function φ defined on an infinite real interval I, Lebesgue integrable on every finite interval contained in I and such that for every $x \in I - \sup\{|M(s)| :\ s \in S\}$ the integral $\int_S \varphi(x + M(s))\sigma(ds)$ exists and (1) holds.

If λ is a root of (2), then the function $x \mapsto e^{\lambda x}, x \in \mathbb{R}$, is a positive solution of (1). We have also the following converse statement.

Theorem. *If* (1) *has a solution with a constant sign, then* (2) *has a real root.*

February 12 (Saturday)
Applications to the probability theory, the aggregation problem and analysis

Tamás Glavosits

The general solution of a functional equation related to the characterizations of bivariate distributions
(joint work with Károly Lajkó)

Seimatsu Narumi used functional equations in his characterization of joint distributions by means of conditional distributions in his paper (On the General Forms of Bivariate Frequency Distributions which are Mathematically Possible when Regression an Variation are Subjected to Limiting Conditions, I., II. Biometria 15 1923 77-88, 209-221) .

Let(X, Y) be an absolutely continuous bivariate random variable. Let us denote the joint, marginal and conditional densities by $f_{(X,Y)}$, f_X, f_Y, $f_{X|Y}$, $f_{Y|X}$, respectively. One can write $f_{(X,Y)}$ in two different ways and obtain the equation

$$f_{X|Y}(x \mid y) f_Y(y) = f_{Y|X}(y \mid x) f_X(x) \qquad (x, y \in \mathbb{R}). \tag{3}$$

Narumi studied joint densities whose conditional densities satisfy

$$f_{X|Y}(x \mid y) = \Psi_1[(x - f_1(y))g_1(y)], \quad f_{Y|X}(y \mid x) = \Psi_2[(y - f_2(x))g_2(x)], \tag{4}$$

where f_1, f_2, g_1, g_2 are given functions and Ψ_1, Ψ_2 are unknown functions.

Narumi investigated (4) for a few particular choices of the given functions f_1, f_2, g_1, g_2. For example, if $f_1, f_2, \frac{1}{g_1}, \frac{1}{g_2}$ are linear functions, he obtained the functional equation

$$G_1\left(\frac{x+1}{y}\right) + F_1(y) = G_2\left(\frac{y+1}{x}\right) + F_2(x) \qquad (x, y \in \mathbb{R}_+) \tag{5}$$

from (3) and (4).

Narumi considered the other case, when f_1, f_2, g_1, g_2 are linear functions, and from (3) and (4) he obtained the functional equation

$$G_1\left(x(y+1)\right) + F_1(y) = G_2\left(y(x+1)\right) + F_2(x) \qquad (x, y \in \mathbb{R}_+). \tag{6}$$

Narumi gave the solutions of the functional equations (5) and (6) in his paper assumed the existence of derivatives of the unknown functions F_1, F_2, G_1, G_2 up to the second or third order.

Here we present the general solutions of the functional equations (5) and (6).

The proof of the general solution of the functional equation (6) give the following

Problem. What is the general solution of the functional equation

$$H(2xy + x) + H(xy + y) = H(2xy + y) + H(xy + x) \qquad (x, y \in \mathbb{R}_+),$$

where $H : \mathbb{R}_+ \to \mathbb{R}$ is the unknown function.

Rafał Kucharski

Generalization of the aggregation problem
(joint work with Maciej Sablik)

We'll deal with determining the aggregation methods for problems of allocation, generalizing results obtained by Aczél, Ng and Wagner. We'll consider the case where the assigned values lie in a vector space and investigate the cases where we admit aggregation of opinions being continuous functions or random variables.

Zita Makó

On the Lipschitz perturbation of monotonic functions
(joint work with Zsolt Páles)

Let $d : I^2 \to I$ be a semimetric. A real valued function f defined on a real interval I is called d-Lipschitz if it satisfies

$$|\ell(x) - \ell(y)| \leq d(x, y)$$

for $x, y \in I$. The main result states that a function $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$ where q is increasing and ℓ is d-Lipschitz if and only if the

$$\sum_{i=1}^{n} \Big(p(s_i) - p(t_i) - d(t_i, s_i)\Big)^+ \leq \sum_{j=1}^{m} \Big(p(v_j) - p(u_j) + d(u_j, v_j)\Big)$$

inequality is fulfilled for all real numbers $t_1 < s_1, \dots, t_n < s_n, u_1 < v_1, \dots, u_m < v_m$ in I which satisfying the condition

$$\sum_{i=1}^{n} 1_{]t_i,s_i]} = \sum_{j=1}^{m} 1_{]u_j,v_j]}.$$