The 3rd International Students' Conference

THE THEORY OF FUNCTIONAL EQUATIONS AND INEQUALITIES AND ITS APPLICATIONS

Szczyrk, Poland

February 3–6, 2007

Karol Baron

Random-valued vector functions and iterative functional equations

Given a probability space $(\Omega, \mathcal{A}, P)$, a Polish space X with the σ-algebra $\mathcal{B}(X)$ of all its Borel subsets and a $(\mathcal{B}(X) \otimes \mathcal{A})$-measurable function $f\colon X \times \Omega \to X$ we consider solutions $\varphi\colon X \to \mathbb{R}$ of

$$(*) \qquad \varphi(x) = \int_\Omega \varphi(f(x,\omega))P(d\omega)\,.$$

Let $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$ denote the product of infinite sequence of the spaces $(\Omega, \mathcal{A}, P)$ and define the iterates $f^n\colon X \times \Omega^\infty \to X$, $n \in \mathbb{N}$, of f by

$$f^1(x, \omega_1, \omega_2, \ldots) = f(x, \omega_1)\,,$$

$$f^{n+1}(x, \omega_1, \omega_2, \ldots) = f(f^n(x, \omega_1, \omega_2, \ldots), \omega_{n+1})\,.$$

Since they are $(\mathcal{B}(X)\otimes\mathcal{A}^\infty)$-measurable, we can define the function $\Pi\colon X\times\mathcal{B}(X) \to [0,1]$ putting

$$\Pi(x,B) = P^\infty(\{\omega \in \Omega^\infty : \begin{array}{l}\text{the sequence } (f^n(x,\omega))_{n\in\mathbb{N}} \text{ converges}\\ \text{in } X \text{ and its limit belongs to } B\}) \,.\end{array}$$

According to [R. Kapica, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 213 (2004), 113-118] for any $B \in \mathcal{B}(X)$ the function $\Pi(\cdot, B)$ is a Borel solution of $(*)$. It turns out that if for a function $S\colon X \to 2^X$ with countable values we have

$$\Pi(x, S(x)) = 1 \quad \text{for } x \in X\,,$$

then any Borel, bounded and continuous at each point of $\bigcup_{x\in X} S(x)$ solution $\varphi\colon X \to \mathbb{R}$ of $(*)$ has the form

$$\varphi(x) = \sum_{y\in S(x)} \Pi(x, \{y\})\varphi(y) \quad \text{for } x \in X\,.$$

Grzegorz Bartosz and Pola Siwek

On a method of improving regularity of solutions of a functional inequality
(work of Michał Lewicki)

Let $I \subset \mathbb{R}$ be an open interval and $M, N \colon I^2 \to I$ be means on I. Let $\varphi \colon I \to \mathbb{R}$ be a solution of the functional equation:

$$\varphi(M(x,y)) + \varphi(N(x,y)) = \varphi(x) + \varphi(y)\,, \quad x, y \in I\,.$$

We give sufficient conditions on means M, N and the function φ which guarantee that for every Lebesgue measurable solution $f \colon I \to \mathbb{R}$ of the functional inequality:

$$f(M(x,y)) + f(N(x,y)) \leq f(x) + f(y)\,, \quad x, y \in I$$

the function $f \circ \varphi^{-1} \colon \varphi(I) \to \mathbb{R}$ is convex.

Włodzimierz Fechner

A joint generalization of a functional inequality and a stability problem

Let X and Y be abelian groups and let K be a subset of Y. Assume that $f\colon X \to Y$ and $\phi\colon X \times X \to Y$ satisfy

$$f(x+y) - f(x) - f(y) - \phi(x,y) \in K\,, \quad x, y \in X.$$

We will provide conditions upon f, ϕ and K which force the representation

$$f = q + A\,,$$

where $q\colon X \to Y$ is a quadratic functional and $A\colon X \to Y$ satisfies

$$A(x+y) - A(x) - A(y) \in cK\,, \quad x, y \in X\,,$$

for some $c > 0$.

In particular, our results cover the case, where K is a real halfline containing zero and the case, where K is a closed ball with the center at the origin in a Banach space.

Żywilla Fechner

The inequality in the change of variables
(work of Władysław Kulpa)

We present a theorem due to Professor Władysław Kulpa (an unpublished result, presented on his lectures at Silesian University), which provides a generalization of the classical theorem on the change of variables in the Lebesgue integral. As a corollary we derive the celebrated Sard's theorem.

Roman Ger

A single equation defining ring homomorphisms

A survey of results concerning the functional equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y)$$

and its miscellaneous alternates will be reported on. The chief concern is to examine whether or not this functional equation is equivalent to the system of two Cauchy equations defining a ring homomorphism.

A similar problem will be discussed in connection with ring derivations and the corresponding equation

$$f(x+y) + f(xy) = f(x) + f(y) + xf(y) + yf(x)$$

along with its several alterations.

Tomasz Kochanek

Stability aspects of number-theoretically additive functions

We study arithmetic mappings $f\colon \mathbb{N} \to \mathbb{R}$ which are almost number-theoretically additive in the following sense:

$$|f(xy) - f(x) - f(y)| \leq \varepsilon \quad \text{for relatively prime } x, y \in \mathbb{N}$$

and with a fixed nonnegative constant ε. We prove that under certain regularity condition on f the above stability assumption implies that f lies near to an number-theoretically additive function, with an error not exceeding ε.

Furthermore, the following condition:

$$\liminf_{x\to\infty}(f(x+1) - f(x)) \geq 0\,,$$

together with the above stability assumption, forces f to be approximated by a logarithm function. This statement generalizes the classical theorems of P. Erdős and A. Mate.

Tomasz Szostok

Functional equations stemming from quadrature rules
(joint work with Barbara Koclęga-Kulpa and Szymon Wąsowicz)

We consider the functional equations of the form

$$F(y) - F(x) = (y - x)\sum_{k=1}^{n} \alpha_k f(\alpha_k x + \beta_k y) \tag{$*$}$$

for functions acting on an integral domain. Equations of this type are used in numerical analysis (for real functions) and are called quadrature rules. Such equations are used for approximate integration however in some cases they are exact. Namely this equation is satisfied by polynomials (with appropriate choice of constants occurring in $(*)$). Moreover the degree of polynomial which satisfies $(*)$ depends on the length of the right hand side of this equation. We present the solutions of this equation in some cases.