Problems and remarks

Roman Ger: An "old" problem

Given a function $f\colon \mathbb{R} \to \mathbb{R}$ assume that the inequality

$$\Delta_h^{n+1} f(x) \geqslant 0,$$

holds true for ℓ_2-almost all pairs $(x, h) \in \mathbb{R} \times (0, \infty)$. Does there exist a function $g\colon \mathbb{R} \to \mathbb{R}$ such that the inequality

$$\Delta_h^{n+1} g(x) \geqslant 0,$$

is satisfied for **all** pairs $(x, h) \in \mathbb{R} \times (0, \infty)$ and

$$\ell_1(\{x \in \mathbb{R} : \ f(x) \neq g(x)\}) = 0?$$

Here Δ_h^p stands for the p-th iterate of the difference operator $\Delta_h \varphi(x) := \varphi(x + h) - \varphi(x)$, whereas ℓ_p denotes the p-dimensional Lebesgue measure, $p \in \mathbb{N}$.

Roughly speaking, the question is whether almost n-convex function has to be almost equal to an n-convex one.

M. Kuczma has shown in [2] that for Jensen convex functions (i.e. for $n = 1$) the answer is positive. For *polynomial functions*, i.e. for solutions of the Fréchet functional equation

$$\Delta_h^{n+1} f(x) = 0,$$

the answer to an analogous question is affirmative as well (an "old" result published in [1]; see also [3]).

In a more general setting, the problem in question carries over to the case where the real line $\mathbb{R}$ is replaced by an Abelian group and the σ-ideals of nullsets in $\mathbb{R}$ and $\mathbb{R}^2$ are replaced by an abstract proper linearly invariant ideal (σ-ideal) $\mathcal{I}$ and the ideal $\Omega(\mathcal{I})$, respectively (see [3] for detailed definitions).

References

[1] R. Ger, *On almost polynomial functions*, Colloquium Mathematicum **24** (1971), 95–101.
[2] M. Kuczma, *Almost convex functions*, Colloquium Mathematicum **21** (1970), 279–284.
[3] M. Kuczma, *An introduction to the theory of functional equations and inequalities*, Birkhäuser Verlag, Basel–Boston–Berlin, 2008.

Włodzimierz Fechner: Which spaces are Hlawka spaces?

Hlawka's inequality:

$$\|x + y\| + \|y + z\| + \|x + z\| \leqslant \|x + y + z\| + \|x\| + \|y\| + \|z\| \tag{1}$$

holds true in each inner product space. Therefore, it is true on the real line and, consequently, on spaces ℓ^1 and $L^1([0, 1])$.

The following example shows that there exist normed linear spaces which are not Hlawka spaces (i.e. in which inequality (1) is not valid). Let us consider the space $\mathbb{R}^3$ with the supremum norm and take $x = (1, 1, -1)$, $y = (1, -1, 1)$ and $z = (-1, 1, 1)$. Then

$$\|x + y\| + \|y + z\| + \|x + z\| = 6$$

whereas

$$\|x + y + z\| + \|z\| + \|y\| + \|z\| = 4.$$

Modifying this example we can obtain something more: if x, y, z are the same as before and the space $\mathbb{R}^3$ is equipped with the norm:

$$\|(t_1, t_2, t_3)\|_p = (|t_1|^p + |t_2|^p + |t_3|^p)^{\frac{1}{p}}$$

then Hlawka inequality (1) is violated for any $p > \log_{1.5} 3 \approx 2.7095$.

L.M. Kelly, D.M. Smiley, M.F. Smiley [1] showed that each two-dimensional real normed space is a Hlawka space. This result (which is elementary and easy to prove) can be also deduced from a (by no means elementary) result of J. Lindenstrauss [2] which states that each two-dimensional real normed linear space E is isomorphically isometric to a subset of $L^1([0, 1])$ (and therefore E is a Hlawka space). Further, using some results from paper J. Lindenstrauss, A. Pełczyński [3] one can deduce that all Banach spaces having the property that its all finite dimensional subspaces can be embedded linearly and isometrically in the space $L^p([0, 1])$, with some $1 \leqslant p \leqslant 2$ are Hlawka spaces (see C.P. Niculescu, L.-E. Persson [6]). Further, H.S. Witsenhausen [7] proved that a finite-dimensional real space with piecewise linear norm is embeddable in L^1 if and only if it is a Hlawka space. However, A. Neyman [5] showed that in general case embeddability in L^1 does not characterize Hlawka spaces.

In book of D.S. Mitrinović [4] the author wrote that (1) is valid on each real normed space (clearly this is not true – see example above). For the proof he refers the reader to [1] (with the proof of (1) in 2-dimensional case only). Several authors refer to this erroneous fact (and they are using it!).

The problems I would like to draw your attention on are:

- for which $p > 1$ the spaces ℓ^p and L^p are Hlawka spaces?
- when arbitrary normed space is a Hlawka space?

References

[1] L.M. Kelly, D.M. Smiley, M.F. Smiley, *Two dimensional spaces are quadilateral spaces*, Amer. Math. Monthly **72** (1965), 753–754.

[2] J. Lindenstrauss, *On the extension of operators with a finite-dimensional range*, Illinois J. Math. **8** (1964), 488–499.

[3] J. Lindenstrauss, A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. **29** (1968), 275–326.

[4] D.S. Mitrinović, *Analytic inequalities*, Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag (1970).

[5] A. Neyman, *Representation of L_p-norms and isometric embeddings in L_p-spaces*, Israel J. Math. **48** (1984), 129–138.

[6] C.P. Niculescu, L.-E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics (Ouvrages de Mathématiques de la SMC 23). New York, NY: Springer. xvi, (2005).

[7] H.S. Witsenhausen, *A support characterization of zonotopes*, Mathematika **25** (1978), 13–16.

Tamas Glavosits

We say that an polygon is a regular polygon, if its sides are congruent with each other and its angles are also congruent with each other. We use the abbreviation n-gon for regular polygons with n-sides, where n is a positive integer. We say that a solid is a Platonic solid, if it is a convex regular polyhedron with congruent regular faces. There are only 5 Platonic solids, while the number of regular polygons is infinite. The concept of Archimedean solid is a generalization of the concept of Platonic solid. The Archimedean solids are semi-regular convex polyhedrons which are composed two or more types of regular polygons meeting in identical vertices.

Problems with solutions

The following problem was presented at the conference *Convexity & Applications* (Iwonicz Zdrój, Polska, September 5–10, 2010).

Problem 1 [The special problem in 2D]. Let T be an arbitrarily fixed equilateral triangle and let S be an arbitrarily fixed square. We would like to find the plane-figure which is the intersection of all the squares that contain T and the common center of which is the same as the center of T, and we would like to find the plane-figure which is the intersection of all the equilateral triangles that contain S and the common center of which is the same as the center of S.

Solution 2 [By Wolfgang Förg-Rob]. In both of the cases investigated above we obtain an equilateral 12-gon such that the vertices of the origin triangle (or square) can be found among the vertices of this 12-gon.

Problem 3 [The general problem in 2D]. Let n, m be fixed positive integers such that $n, m \geqslant 3$. Take an arbitrarily fixed n-gon F. We want to know what the intersection of all the m-gons that contain the fixed n-gon and the common center of these m-gons is the same as the center of the original F.

Solution 4. The method applied by Wolfgang Förg-Rob can be used in this more general situation. The intersection of all m-gons having the required properties an $\operatorname{lcm}(m, n)$-gon such that the vertices of the origin n-gon F can be found among the vertices of this $\operatorname{lcm}(m, n)$-gon.

Open problems

Problem 5. The two problems above (namely the special and the general problem in 2D) can be formulated in 3D such that in special problem we can use tetrahedron or cube instead of equilateral triangle or square and in the general problem we can use use Platonic solids or Archimedean solids (or any other type semi-symmetric convex solids) instead of n-gons. These problems can be extended into higher dimensional spaces.

Tomasz Szostok

A function f is called strongly midconvex (or Jensen convex) with modulus equal to 1, if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} - \frac{1}{4}|x-y|^2. \tag{1}$$

It can be easily proved (see [1]) that if f is midconvex, then it satisfies the inequality

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y) - t(1-t)|x-y|^2, \tag{2}$$

for all dyadic numbers $t \in (0,1)$. If additionally f is continuous, then this inequality is satisfied for all t, i.e. f is strongly convex.

Inequality (1) may be generalized in the following way:

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} - \frac{1}{2^p}|x-y|^p. \tag{3}$$

But then, the step leading to an inequality similar to (2) is less obvious. For example, it can be shown that if f satisfies (3), then

$$f(tx+(1-t)y) \leqslant tf(x) + (1-t)f(y) - (t-t^p)|x-y|^p$$

for all t of the shape $t = 1/2^k$. Thus, the problem is to find a function $h(t)$ such that every function satisfying (3) satisfies

$$f(tx+(1-t)y) \leqslant tf(x) + (1-t)f(y) - h(t)|x-y|^p$$

for every dyadic number $t \in (0,1)$.

Reference

[1] A. Azócar, J. Giménez, K. Nikodem, J.L. Sánchez, *On strongly midconvex functions*, Opuscula Mathematica **31**, no. 1 (2011), 15–26.

Mateusz Jurczyński

The integral Hadamard inequality can be stated as follows:

$$\left|\int_{\mathbb{R}^n} |Df(x)|^{p-n} J(x,f)dx\right| \leqslant \lambda \int_{\mathbb{R}^n} |Df(x)|^p dx.$$

It is natural to ask what is the best λ such that this inequality holds true for given p, n (and $f \in \mathcal{C}_0^\infty$). Through study of quasiconvex functions, Tadeusz Iwaniec proposed the following conjecture.

Conjecture. The best constant λ in the integral Hadamard inequality for $n < 2p$ is

$$\lambda_p(n) = \left|\frac{n}{p} - 1\right|.$$

It is still an open problem.

Reference

[1] A. Kałamajska, *Quasiconvexity and open problems of functional analysis*, Wiad. Mat. **46**(1) (2010), 17–26.